with

$$F_1^{mn} = e^{\alpha\zeta}\left[\cos\beta\zeta\sum_{i=0}^{\infty}\Re\left(Y_i^1\right)\zeta^i - \sin\beta\zeta\sum_{i=0}^{\infty}\Im\left(Y_i^1\right)\zeta^i\right]$$

$$F_2^{mn} = e^{\alpha\zeta}\left[\sin\beta\zeta\sum_{i=0}^{\infty}\Re\left(Y_i^1\right)\zeta^i + \cos\beta\zeta\sum_{i=0}^{\infty}\Im\left(Y_i^1\right)\zeta^i\right]$$

The terms $\Re$ and $\Im$ indicate the real and imaginary parts of a complex number. Thus the complete solution X_{mn} is

$$X_{mn}(\zeta) = \sum_{j=1}^{8} F_j^{mn}(\zeta)C_j^{mn} + \sum_{j=1}^{2} G_j^{mn}(\zeta)A_j^{mn}$$
$$\text{with}\quad G_j^{mn}(\zeta) = e^{\rho_j\zeta}\left(\sum_{i=0}^{\infty} Z_i^j\zeta^j\right) \tag{14}$$

where $F_j^{mn}(\zeta)$ is given by Eq. (13) and C_j^{mn} are real constants. The various Z_i^j are obtained by a recursive relation, which is omitted here for brevity. The infinite power series in $F_j^{mn}(\zeta)$ and $G_p^{mn}(\zeta)$ are truncated such that the contribution of the first neglected term is less than 10^{-10}.

The $2L$ constants $(A_j^{mn})^{(k)}$ for L layers are determined from the $2L$ thermal conditions (4) and (6). The $8L$ constants $(C_j^{mn})^{(k)}$ for L layers and $L_p + L_a$ unknown extraneous surface charge densities $\sigma_{i_{mn}}$ and $\tau_{q_{mn}}$ are obtained from $8L + L_p + L_a$ conditions (3), (5), and (7).

III. Numerical Results and Conclusions

Consider a shell made of cross-ply graphite-epoxy laminate $[0/90/0]_s$ and a layer of lead zirconate titanate (PZT)-5A of thickness $h/10$, bonded to its outer surface. The orientation of the fibers is given relative to the θ direction. All plies of the substrate have equal thickness. The material properties are selected as in Ref. 6. The interface of the piezoelectric layer with the substrate is grounded. Loads with the following nonzero P_i, T_i, or ϕ_i are considered: 1) $P_2 = p_0\cos 4\pi\xi_1 \sin\pi\xi_2$, 2) $T_2 = T_0\cos 4\pi\xi_1\sin\pi\xi_2$, and 3) $\phi_2 = \phi_0\cos 4\pi\xi_1\sin\pi\xi_2$. The results for the three cases are nondimensionalized as follows with $\bar{a} = a/R$, $S = R/h$, $d_T = 374\times 10^{-12}\ \mathrm{CN}^{-1}$, $\alpha_T = 22.5\times 10^{-6}\ \mathrm{K}^{-1}$, $E_T = 10.3$ GPa:

1) $(\bar{u}, \bar{w}) = 10(u, \bar{a}w)E_T/hS^3p_0, \quad (\bar{\sigma}_\theta, \bar{\sigma}_z) = (\sigma_\theta, \sigma_z)/S^2p_0$

2) $\hat{u} = 100u/h\alpha_T S^2 T_0, \quad \hat{\sigma}_\theta = \sigma_\theta/\alpha_T E_T T_0$

3) $\tilde{u} = 10u/S^2 d_T\phi_0, \quad \tilde{\sigma}_\theta = \sigma_\theta h/E_T d_T\phi_0$

The effect of the length parameter $\bar{a}$ is studied for thick ($S = 4$) and thin ($S = 40$) shells.

The through-the-thickness distributions of some entities for the pressure load case 1 are shown in Figs. 1 and 2. It is observed from Fig. 1 that the deflection $\bar{u}$ is almost uniform for thin shells with $S = 40$. The variation of the axial displacement $\bar{w}$ across the thickness is linear for the thin shells and is piecewise linear for thick shells. The distributions of the predominant stresses $\bar{\sigma}_\theta$ and $\bar{\sigma}_z$, shown in Fig. 2, reveal that the relative increase of $\bar{\sigma}_\theta$ with $\bar{a}$ is less for the thick shell compared with the thin shell.

The results for thermal load case 2 are given in Fig. 3. The distribution of $\hat{u}$ across the whole thickness is almost uniform for thin shells with $S = 40$. For thick shells, the distribution of $\hat{u}$ and $\hat{\sigma}_\theta$ across the elastic substrate is nonlinear. The distributions of $\tilde{u}$ and $\tilde{\sigma}_\theta$ are presented in Fig. 4 for the potential load case 3. The variation of $\tilde{u}$ in the piezoelectric layer is linear. The variation of $\tilde{\sigma}_\theta$ for thick shells is relatively less nonlinear compared with thermal load case 2.

It is inferred from the results that the displacements and the predominant normal stresses in the substrate are significantly affected by the radius to thickness ratio but the nature of their through-the-thickness distributions is not affected much by the length-to-radius ratio for both thick and thin shells.

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Effective Mass Sensitivities for Systems with Repeated Eigenvalues

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Introduction

THE effective modal mass, commonly referred to as effective mass, is quite important in characterizing the dynamical behavior of a base driven structure because it allows the reduction of complex structures to equivalent spring-mass systems[1,2] and the identification of the modes that can be significantly excited through the interface by the base motion. These modes are called target modes, and they have to be well correlated to the experimental ones to obtain a test verified finite element (FE) model.[3] The effective mass sensitivities, on the other hand, can be used in optimization problems such as that of finding the optimal position of an extra payload to be added[4,5] or that of the minimization of errors between experimental and numerical effective masses. In this work the calculation of effective mass sensitivities has been generalized to the case of repeated eigenvalues.

Theory

The definition of effective mass matrix is[2,6]

$$\boldsymbol{M}_{\mathrm{eff}_i\,(r\times r)} = \left(\boldsymbol{X}_R^T\boldsymbol{M}\boldsymbol{x}_i\right)\left(\boldsymbol{x}_i^T\boldsymbol{M}\boldsymbol{X}_R\right) \tag{1}$$

where r is the number of rigid body modes of the structure, $\boldsymbol{X}_R$ is the rigid body mode matrix, $\boldsymbol{x}_i$ is the ith eigenvector of the eigenvalue problem

$$(\boldsymbol{K} - \lambda_i\boldsymbol{M})\boldsymbol{x}_i = 0 \tag{2}$$

$\boldsymbol{M}$ and $\boldsymbol{K}$ are the mass and stiffness matrices of the FE model of the structure, and λ_i is the ith eigenvalue. By differentiating Eq. (1)

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with respect to a design variable α_k, one obtains the expression of the effective mass sensitivity matrix

$$\frac{\partial \boldsymbol{M}_{\mathrm{eff}_i}}{\partial \alpha_k} = \left(\boldsymbol{X}_R^T \frac{\partial \boldsymbol{M}}{\partial \alpha_k} \boldsymbol{x}_i\right)\left(\boldsymbol{x}_i^T \boldsymbol{M} \boldsymbol{X}_R\right) + \left(\boldsymbol{X}_R^T \boldsymbol{M} \frac{\partial \boldsymbol{x}_i}{\partial \alpha_k}\right)\left(\boldsymbol{x}_i^T \boldsymbol{M} \boldsymbol{X}_R\right) + \left(\boldsymbol{X}_R^T \boldsymbol{M} \boldsymbol{x}_i\right)\left(\frac{\partial \boldsymbol{x}_i^T}{\partial \alpha_k} \boldsymbol{M} \boldsymbol{X}_R\right) + \left(\boldsymbol{X}_R^T \boldsymbol{M} \boldsymbol{x}_i\right)\left(\boldsymbol{x}_i^T \frac{\partial \boldsymbol{M}}{\partial \alpha_k} \boldsymbol{X}_R\right) \tag{3}$$

In this expression the eigenvector derivatives are present for which the following considerations are essential.

The eigenvalue problem (2), for the case of unrepeated roots, yields n distinct eigenvalues λ_i and ∞^n (meaning n arbitrary constants) eigenvectors $\boldsymbol{x}_i$. The vectors $\boldsymbol{x}_i$, as well known, are orthogonal with respect to the mass matrix. By introducing the normality condition $\boldsymbol{x}_i^T \boldsymbol{M} \boldsymbol{x}_i = 1$ a unique solution is obtained, i.e., one orthonormal set of n eigenvectors. In the case of one repeated root of multiplicity m, the eigenvalues are as follows:

$$\lambda_1, \lambda_2, \ldots, \lambda_l, \lambda_{l+1}, \lambda_{l+2}, \ldots, \lambda_{l+m}, \ldots, \lambda_n$$

$$\text{with} \qquad \lambda_{l+1} = \lambda_{l+2} = \cdots = \lambda_{l+m} = \lambda \tag{4}$$

Associated with the repeated eigenvalues there is a subspace, orthogonal to the other $n - m$ eigenvectors. In this subspace one can arbitrarily define ∞^{m^2} sets of m eigenvectors (m independent components for each of the m eigenvectors). To reduce the arbitrariness, the condition of orthogonality to the set of eigenvectors corresponding to the repeated root can be imposed:

$$\boldsymbol{x}_i^T \boldsymbol{M} \boldsymbol{x}_j = 0, \qquad i, j = l+1, \ldots, l+m \quad \text{and} \quad i \neq j \tag{5}$$

These are

$$\binom{m}{2} = \frac{m!}{2!(m-2)!} = \frac{m(m-1)}{2} \tag{6}$$

relations and express the number of all of the possible combinations of two among the m eigenvectors. Therefore, the possible sets of m eigenvectors are reduced to ∞^μ with

$$\mu = m^2 - \binom{m}{2} = \frac{m^2 + m}{2} \tag{7}$$

It is easy to verify that μ is also equal to

$$m + \binom{m}{2} \tag{8}$$

i.e., m arbitrary constants for the length of each eigenvector and $m(m-1)/2$ for the orientation of the orthogonal set of m eigenvectors. By further imposing the normality condition

$$\boldsymbol{x}_i^T \boldsymbol{M} \boldsymbol{x}_i = 1, \qquad i = l+1, \ldots, l+m \tag{9}$$

m more relations are introduced. Consequently, the number of sets of the m orthonormal eigenvectors is reduced to ∞^ν, where $\nu = \mu - m$. Therefore, even by extending the orthogonality condition to the case of repeated roots and by normalizing to unity the eigenvectors, the indeterminacy is still present. It can be eliminated by choosing the only set that makes the differentiation with respect to the design parameter meaningful. Such a set is called adjacent. To find the adjacent set $\boldsymbol{Z}$, an orthogonal transformation $\boldsymbol{\Gamma}$ is introduced:

$$\boldsymbol{Z}_{(n \times m)} = \boldsymbol{X}_{(n \times m)} \boldsymbol{\Gamma}_{(m \times m)} \tag{10}$$

which can be found by a procedure proposed in Ref. 7 or, in improved form, in Ref. 8. The matrix $\boldsymbol{\Gamma}$ and the m eigenvalue derivatives Λ' associated with the repeated root $\boldsymbol{\lambda}$ are given by the eigenvalue problem

$$\boldsymbol{D}_{(m \times m)} \boldsymbol{\Gamma}_{(m \times m)} = \boldsymbol{\Gamma} \Lambda'_{(m \times m)} \tag{11}$$

where

$$\boldsymbol{D} = \boldsymbol{X}^T (\boldsymbol{K}' - \lambda \boldsymbol{M}') \boldsymbol{X} \tag{12}$$

is a matrix connected to the system matrix derivatives

$$\boldsymbol{K}' = \frac{\partial \boldsymbol{K}}{\partial \alpha_k}, \qquad \boldsymbol{M}' = \frac{\partial \boldsymbol{M}}{\partial \alpha_k} \tag{13}$$

Following the Dailey[8] procedure, it was possible to determine the eigenvector set $\boldsymbol{Z}$ and its derivative $\boldsymbol{Z}'$ whose columns, substituted for $\boldsymbol{x}_i$ and $\partial \boldsymbol{x}_i / \partial \alpha_k$, respectively, into Eq. (3), allowed the effective mass sensitivities to be found.

Numerical Results

An FE model of an aluminum beam, with two concentrated masses of 1 kg each, was considered (Fig. 1). A very high (even unrealistic) moment of inertia was chosen to have the bending and the extensional modes in mixed order. In spite of its simplicity, the chosen example was quite general because it contained singular degrees of freedom (DOF) (rotation about x axis, i.e., torsion) and both coincident (bending modes) and distinct eigenvalues (extensional modes) (Tables 1 and 2). In the results reported, the design variable considered is the second diagonal element of the mass matrix, i.e., the first mass in the y direction. Analogous results were obtained for the other diagonal mass matrix elements.

Different approaches are considered. The first one is exact and is based on an algorithm reported in Ref. 8. The second one has been applied by using a procedure developed in Ref. 6, which is applicable in the case of noncoincident eigenvalues and is based on a modal expansion of the expression of the eigenvector derivative first proposed in Ref. 9. (This procedure is exact when a complete modal model is used, as in the example reported in this work.) If one lets the coincident eigenvalues be split by introducing a very small perturbation in the FE model in the same direction of the wanted derivative, then that procedure can be applied. In the example, the perturbation was 10^{-6} kg. This practically infinitesimal perturbation had the effect of rotating the eigenvectors by a suitable finite

Table 1 Eigenvalues of FE cantilever beam

i	λ_i	f_i, Hz	Mode type
1	1.049634E+05	5.156310E+01	1st bending in xy plane
2	1.049634E+05	5.156310E+01	1st bending in xz plane
3	1.849096E+06	2.164212E+02	1st extensional
4	4.315776E+06	3.306356E+02	2nd bending in xy plane
5	4.315776E+06	3.306356E+02	2nd bending in xz plane
6	1.219350E+07	5.557563E+02	2nd extensional

Table 2 Eigenvectors of FE cantilever beam

DOF	$\boldsymbol{x}_1$	$\boldsymbol{x}_2$	$\boldsymbol{x}_3$	$\boldsymbol{x}_4$	$\boldsymbol{x}_5$	$\boldsymbol{x}_6$
x_1	0	0	0.4990	0	0	0.7590
y_1	−0.289465	0	0	0.8610	0	0
z_1	0	0.289465	0	0	0.8610	0
x_2	0	0	0.7945	0	0	−0.5223
y_2	−0.901283	0	0	−0.3030	0	0
z_2	0	0.901283	0	0	−0.3030	0

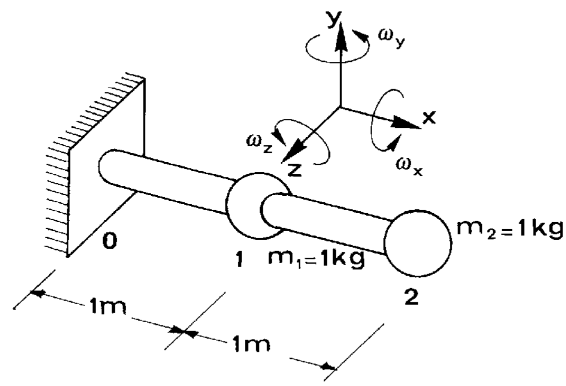

Fig. 1 Beam of two elements with two concentrated masses: E = 70 GPa, ρ = 2700 kg/m^3, A = 7.85E–5 m^2, and I = 4.91E–6 m^4.

Table 3 Eigenvalue sensitivities λ_i'

	1st approach	2nd approach	3rd approach
λ_1'	−8.7949159E+3	−8.7949148E+3	−8.828E+3
λ_2', λ_3'	0	0	0
λ_4'	−3.1990803E+3	−3.199075E+6	3.2E+6
λ_5', λ_6'	0	0	0

Table 4 Derivative of the first eigenvector

DOF	1st approach	2nd approach	3rd approach
x_1	0	0	0
y_1	6.7786668E−3	6.7786673E−3	6.78E−3
z_1	0	−2.8604516E−5	−2.5044E−7
ω_{x1}	0	0	0
ω_{y1}	0	5.0429107E−5	−2.6053E−7
ω_{z1}	1.9894528E−2	1.9894527E−2	1.9898E−2
x_2	0	0	0
y_2	3.9641898E−2	3.9641895E−2	3.9648E−2
z_2	0	−8.9063400E−5	−7.7978E−7
ω_{x2}	0	0	0
ω_{y2}	0	6.5473773E−5	5.7324E−7
ω_{z2}	3.9347583E−2	3.9347579E−2	3.9381E−2

Table 5 Effective mass sensitivities

Approach	x	y	z	ω_x	ω_y	ω_z	Mode
1st	0	6.3989737E−1	0	0	0	9.0761532E−1	1st
1st	0	3.6010263E−1	0	0	0	9.2384676E−2	4th
2nd	0	6.3989745E−1	0	0	0	9.0761535E−1	1st
2nd	0	3.6010255E−1	0	0	0	9.2384646E−2	4th
3rd	0	6.4001E−1	0	0	0	9.0777E−1	1st
3rd	0	3.60158E−1	0	0	0	9.2398E−2	4th

amount that eliminates the indeterminacy described in the preceding section. The third approach is the well-known finite difference method, which can be applied by taking a second perturbation (in the example, 10^{-4} kg). Neither the second nor the third approach is exact because the evaluation of the derivative was not performed on the FE model of the given structure but on a perturbed version of it. Furthermore, the third approach has introduced a finite difference approximation. In Tables 3 and 4 the eigenvalue sensitivities and the sensitivity of the first eigenvector, respectively, are reported. Similar results have been obtained for the other eigenvectors. In Table 5 the diagonals of the effective mass matrix sensitivity for the first and fourth modes are reported. All of these results have shown that the second approach had excellent agreement with the exact solution, at least for eigenvalue and effective mass sensitivities, whereas the third one shows acceptable results.

Another possible approach is as follows: 1) determine λ and $\boldsymbol{X}$ from the given structure, 2) determine $\tilde{\boldsymbol{X}}$, $\tilde{\boldsymbol{M}}$, and $\tilde{\boldsymbol{K}}$ from the perturbed one, 3) calculate $\boldsymbol{D}$ from Eq. (12) rewritten at finite difference

$$\boldsymbol{D} = \boldsymbol{X}^T \left(\frac{\tilde{\boldsymbol{K}} - \boldsymbol{K}}{\Delta M_k} - \lambda \frac{\tilde{\boldsymbol{M}} - \boldsymbol{M}}{\Delta M_k} \right) \boldsymbol{X} \tag{14}$$

and 4) find the eigenvalue derivative λ' and $\boldsymbol{\Gamma}$ from Eq. (11). Once $\boldsymbol{\Gamma}$ is determined, one can calculate the eigenvector derivatives with the finite difference approximation

$$\frac{\partial \boldsymbol{X}_{l+i}}{\partial M_k} \approx \frac{\tilde{\boldsymbol{X}}_{l+i} - \boldsymbol{X}_{l+i}\boldsymbol{\Gamma}}{\Delta M_k} = \frac{\tilde{\boldsymbol{X}}_{l+i} - \boldsymbol{Z}_i}{\Delta M_k} \qquad i = 1, \ldots, m \tag{15}$$

The results obtained were very close to the results of the third approach.

Conclusions

A generalization for the calculation of effective mass sensitivities to the case of coincident eigenvalues has been proposed. The results obtained with the exact approach and an approach based on a suitable perturbation, introduced to eliminate the eigenvalue multiplicity, are in excellent agreement. The two approaches based on the finite difference approximation are still in good agreement, but as expected, the elimination of significant digits does affect the precision.

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Modal Data Are Insufficient for Identification of Both Mass and Stiffness Matrices

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Introduction

IN the technical literature, there have appeared, and continue to appear, many papers in which the researchers propose to identify simultaneously both the mass and stiffness matrices of a dynamic structure by applying only modal measured data. However, it can be shown that even full modal data are insufficient for the identification of both the mass and the stiffness matrices.[1,2] In Refs. 3 and 4, it was proposed to use the measured mode shapes as a reference basis in the process of correction of the mass and the stiffness matrices of a structure. The problem is that the mode shapes are not uniquely defined. Any mode shape can be multiplied by a nonzero constant without changing its physical meaning. In Ref. 5, Huang and Craig, who dealt with a six-degree-of-freedom structure, wrote, "It should be noted that even when all six modes are used, the correct values of the mass and stiffness matrices cannot be obtained." The reason for this is that, for given mode shapes and natural frequencies, the mass and stiffness matrices are not uniquely defined. We will now show that the same mode shapes and natural frequencies can be obtained for an infinite number of different pairs of stiffness and mass matrices.

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